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FREE VIBRATION OF PLATES BY THE HIGH ACCURACY QUADRATURE ELEMENT METHOD

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In this paper, a highly accurate and rapidly converging hybrid approach is presented for the Quadrature Element Method (QEM) solution of plate free vibration problems. The hybrid QEM essentially consists of a collocation method in conjunction with a Galerkin finite element technique, to combine the high accuracy of the Differential Quadrature Method (DQM) for the efficient solution of differential equations with the generality of the finite element formulation. This results in superior accuracy with fewer degrees of freedom than conventional FEM or FDM. A series of numerical tests is conducted to assess the performance of the quadrature plate element in free vibration problems. Anisotropic and stepped thickness plates are investigated as well as mixed boundary conditions and point supports at the edges. In all cases, the results obtained are quite accurate.

1. INTRODUCTION

The Differential Quadrature Method (DQM) [1] has been used in the past by various researchers (see, e.g., references [2–4]) for the efficient treatment of linear and non-linear static and dynamic structural analysis problems. All of the analyses yielded good to excellent results for only a few discrete points due to the use of the high order global basis functions in the computational domain. However, difficulties arise from using continuous basis functions in real-life structural analysis [5]. To alleviate the lack of versatility and the limitations of existing high order series type approximation methods, a 49-degree-of-freedom (DOF) quadrature plate element, developed in reference [6], is used in free vibration problems in this study. One may refer to reference [6] for a more detailed description of the QEM.

2. FORMULATION OF THE QUADRATURE PLATE ELEMENT

The quadrature plate element is closely related to the serendipity Lagrangian element, but with internal points and using basis functions of a higher order. Numerical integration and differentiation or so-called "quadrature" procedures are used extensively in the element formulation to circumvent the problems caused by using high order basis functions. C^0 and C^1 inter-element compatibilities are met exactly for the mid-surface, while the other, C^2 or even C^3, compatibilities are closely approximated at each inter-element boundary by the use of moderately high order basis functions. The 25-node rectangular element has 49 DOF with four corner nodal points, and is shown in Figure 1. The displacement field of the 49-DOF quadrature plate element is expressed in terms of polynomial type basis functions, such that it can be assumed as

$$w(x, y) = \sum_{i=1,5,9,13} [N_{i1}w_i + N_{i2}(\partial w/\partial x)_i + N_{i3}(\partial w/\partial y)_i + N_{i4}(\partial^2 w/\partial x\,\partial y)_i]$$

$$+ \sum_{i=2,3,4,10,11,12} [N_{i1}w_i + N_{i2}(\partial w/\partial y)_i] + \sum_{i=6,7,8,14,15,16} [N_{i1}w_i + N_{i2}(\partial w/\partial x)_i]$$

$$+ \sum_{i=17\text{–}25} [N_{i1}w_i] = \lfloor N \rfloor \{w\}, \tag{1}$$

where N_{ij} are the corresponding basis functions, which can be determined from the specified collocation points. Also, w_i, $(\partial w/\partial x)_i$, $(\partial w/\partial y)_i$ and $(\partial^2 w/\partial x\,\partial y)_i$ are the local degrees of freedom associated with node i.

3. FREE VIBRATION QUADRATURE PLATE ELEMENT

Similarly to the static plate element developed in reference [6], an extension of the method is applied to the formulation of plate free vibration models. The quadrature plate element is again derived based on the discrete Kirchhoff assumptions. Therefore, the governing equation of an isotropic thin plate in small deflection free vibration is given by

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \rho h \frac{\partial^2 w}{\partial t^2}. \tag{2}$$

As stated earlier, the 25-node rectangular element based on these assumptions has 49 DOF. Here, the consistent mass matrix can be obtained from

$$[M] = \int_A \lfloor N \rfloor^{\mathrm{T}} [\rho h] \lfloor N \rfloor \,\mathrm{d}A. \tag{3}$$

Therefore, the plate free vibration governing equation can be written in matrix form as

$$([K_s] - \lambda^2 [M_s])\{w_s\} = \{0\}, \tag{4}$$

in which λ is defined as the frequency parameter, and a subscript s represents the whole discretized system.

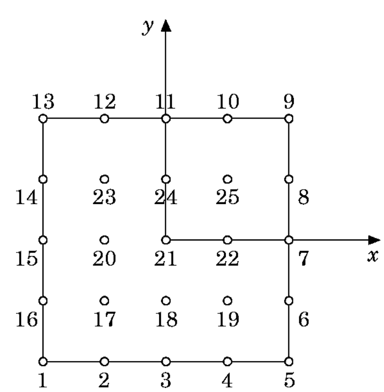

Figure 1. The nodal configuration of a quadrature plate element.

TABLE 1

Convergence study of frequency parameter λ of simply supported square isotropic plate ($\lambda^2 = \rho h \omega^2 a^4/D$), exact: $\omega_{mn} = \pi^2 (D/\rho h)^{1/2} (m^2/a^2 + n^2/b^2)$, with (m, n) being mode numbers

Mode	Mesh (1 × 1)	(2 × 2)	(3 × 3)	Exact
1	19·7392	19·7392	19·7392	19·7392
2	49·4908	49·3480	49·3480	49·3480
3	79·1667	78·9568	78·9568	78·9568
4	100·117	98·7106	98·6961	98·6960
5	129·612	128·317	128·305	128·305
6	—	168·423	167·792	167·783
7	179·729	177·671	177·653	177·653
8	—	197·963	197·400	197·392

4. NUMERICAL APPLICATIONS

The overall stiffness and mass matrices in equation (4) are obtained by assembly procedures as used in the FEM. Various boundary conditions are investigated in the following study. For the QEM, the domain decomposition is achieved by using Galerkin finite element techniques; therefore, inter-element C^2 and C^3 compatibility conditions are not enforced here.

4.1. CONVERGENCE AND COMPARISON STUDIES

For isotropic plates with simple homogeneous boundary conditions on each side, a case with all sides simply supported is investigated, since exact solutions are readily available for direct comparison. The convergence characteristics of the natural frequencies are shown in Table 1. Element meshes of 1×1 to 3×3 for an isotropic square plate are used for the analysis. It can be observed that convergence of the natural frequencies up to at least four significant figures is obtained for the lowest eight modes. In Table 2, the convergence characteristics of the frequency parameter of a clamped square plate are shown. The pattern of convergence achieved for the clamped boundary conditions is similar to that for the simply supported case.

TABLE 2

Convergence study of frequency parameter λ of clamped square isotropic plate ($\lambda^2 = \rho h \omega^2 a^4/D$)

Mode	Mesh (1 × 1)	(2 × 2)	(3 × 3)
1	35·9900	35·9858	35·6852
2	74·1843	73·3968	73·3942
3	108·591	108·227	108·218
4	137·293	131·604	131·582
5	138·070	132·230	132·207
6	168·819	165·027	165·005
7	—	212·088	210·547
8	224·178	220·178	220·043

TABLE 3

Comparison study of frequency parameters λ of clamped square anisotropic plate ($\lambda^2 = \rho h \omega^2 a^4 / D_0$, $D_0 = E_1 h^3/(12(1 - \nu_{12}\nu_{21}))$)

Orientation, θ (degrees)	Solution	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
0	DQM [4]	23·97	31·15	46·38	62·78	—
	Ritz [7]	23·97	31·15	46·41	62·77	—
	QEM (2 × 2)	23·97	31·15	46·42	62·77	67·20
	QEM (3 × 3)	23·97	31·15	46·41	62·77	67·20
15	DQM	23·09	31·51	47·62	59·45	—
	Ritz	23·10	31·52	47·65	59·46	—
	QEM (2 × 2)	23·09	31·51	47·64	59·45	65·78
	QEM (3 × 3)	23·09	31·51	47·64	59·45	65·77
30	DQM	21·33	33·14	50·63	51·79	—
	Ritz	21·35	33·18	50·72	51·87	—
	QEM (2 × 2)	21·34	33·14	50·64	51·82	71·62
	QEM (3 × 3)	21·33	33·14	50·63	51·79	71·25
45	DQM	20·49	34·96	46·85	52·04	—
	Ritz	20·51	35·01	47·07	52·21	—
	QEM (2 × 2)	20·49	34·96	46·92	52·06	69·99
	QEM (3 × 3)	20·49	34·96	46·86	52·04	69·88

4.2. FREE VIBRATION ANALYSIS OF ANISOTROPIC PLATES

For the free vibration of anisotropic plates, the governing differential equation becomes

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4}{\partial x^3\,\partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} = \rho h\frac{\partial^2 w}{\partial t^2}. \quad (5)$$

The formulation for this problem is similar to that for the isotropic case. However, the plate stiffness matrix of the isotropic plate is replaced by

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} = \frac{h^3}{12}\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}, \quad (6)$$

where

$$\bar{Q}_{11} = Q_{11}\alpha^4 + 2(Q_{12} + 2Q_{66})\alpha^2\mu^2 + Q_{22}\mu^4,$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\alpha^2\mu^2 + Q_{12}(\alpha^4 + \mu^4),$$

$$\bar{Q}_{22} = Q_{11}\mu^4 + 2(Q_{12} + 2Q_{66})\alpha^2\mu^2 + Q_{22}\alpha^4,$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\alpha^3\mu + (Q_{12} - Q_{22} + 2Q_{66})\alpha\mu^3,$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\alpha\mu^3 + (Q_{12} - Q_{22} + 2Q_{66})\alpha^3\mu,$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\alpha^2\mu^2 + Q_{66}(\alpha^4 + \mu^4), \quad (7)$$

and where $\alpha = \cos\theta$, $\mu = \sin\theta$ and

$$Q_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \qquad Q_{12} = \frac{\nu_{21}E_2}{1 - \nu_{12}\nu_{21}},$$

$$Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \qquad Q_{66} = G_{12}, \qquad \nu_{21}E_1 = \nu_{12}E_2. \tag{8}$$

First direct comparison, natural frequencies are analyzed by the QEM for clamped square plates composed of an orthotropic material with the principal material axis at θ degrees from the x-axis. The specific material properties are $E_1/E_2 = 10$, $G_{12}/E_2 = 0{\cdot}25$ and $\nu_{12} = 0{\cdot}3$.

Because of the presence of the D_{16} and D_{26} odd derivative terms, the numerical approximation will converge more slowly than it does for the isotropic case. Approximate methods for the vibration analysis of anisotropic plates subject to simple boundary conditions are numerous. Amongst these methods, the high order approximation DQM provides a very compact and very efficient procedure for clamped anisotropic cases [4]. Numerical results obtained by the QEM and comparisons to the DQM and the Ritz method are given in Table 3. Close agreement is observed in all cases. It seems that the QEM again provides excellent convergence and efficiency in the application to anisotropic plate free vibration analyses. Although the DQM provides a simple and efficient means of solving some cases, it has limitations and restrictions in other applications. For instance, rapid convergence can be achieved only when the field variable and its derivative(s) are continuous in the computational domain which has to be bounded by simple boundary conditions, because of the use of series type global basis functions. In the following sections, the QEM will be employed to analyze a family of problems, such as mixed boundary conditions, stepped thickness plates, ordinary cantilever plates, and cantilever plates with point supports at the free edges, which are difficult for the application of the DQM. Compared with the DQM, the QEM provides a more versatile scheme in these applications.

4.3. APPLICATION TO MIXED BOUNDARY CONDITIONS

A high order approximation will deliver good convergence only under the conditions that the field variables in the computational domain are continuous and bounded by simple boundaries. The presence of singular points on the boundary will inflict relatively heavier losses on a high order approximation than on low order fine mesh numerical schemes because of the coarse mesh used. Although, in reference [8], Chebyshev grid spacing is

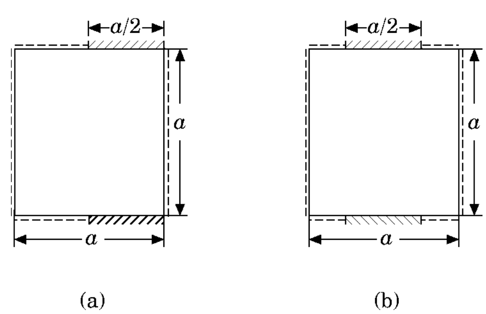

Figure 2. Mixed boundary conditions for square plates: (a) case 1; (b) case 2. ////, clamped; – – – –, simply supported.

TABLE 4

Frequency parameters λ of isotropic square plates with mixed boundary conditions ($\lambda^2 = \rho h \omega^2 a^4/D$)

Solution	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
Ota [9]	25·5	—	—	—	—
Fan [10]	26·37	52·23	61·78	—	—
QEM (2 × 2)	26·29	52·13	61·45	88·06	100·6
QEM (3 × 3)	26·22	52·17	61·30	88·17	100·6
QEM (4 × 4)	26·02	52·13	60·80	88·13	100·6
Ota [9]	28·3	—	—	—	—
Fan [10]	28·65	54·00	68·58	—	—
Narita [11]	28·44	53·49	67·85	90·50	100·6
QEM (2 × 2)	28·67	54·06	68·51	92·27	101·2
QEM (3 × 3)	28·62	53·82	68·35	91·53	100·8
QEM (4 × 4)	28·49	53·58	68·00	90·78	100·6

effective in curing corner singularities, it is not effective for mixed boundary conditions due to the fixed nature of the collocation points.

In this section, two mixed boundary condition cases are presented (Figure 2) to illustrate the accuracy of the QEM. The boundary conditions and resulting fundamental frequency parameters are presented in Table 4 with comparisons to other series type methods or fine mesh numerical methods. Similar mode shapes are found for the two cases. It should be noted that the coarse mesh models show very good agreement with the comparison results from refined mesh analysis. Since the domain decomposition of the QEM can be chosen to provide high resolution in the critical regions adjacent to the singular points or where large gradients of field variables will occur, the QEM will be more versatile in applications than other series type high order numerical schemes.

4.4. APPLICATION TO STEPPED THICKNESS PLATES

A stepped thickness plate is considered in this section. The rectangular plate is thin, isotropic, fully simply supported and singly stepped, as shown in Figure 3.

This problem was solved by Chopra [12]; however, Warburton [13] pointed out that the interface compatibility conditions and, thus, the given numerical results were slightly in error. Later, Yuan *et al*. [14] recalculated the Levy type analytical solution and proposed a Rayleigh–Ritz method for comparison. The numerical results for the QEM, compared with the correct Levy type solutions and the Rayleigh–Ritz results obtained by Yuan, are given in Table 5. Here, the frequency parameters calculated by the QEM for a square plate

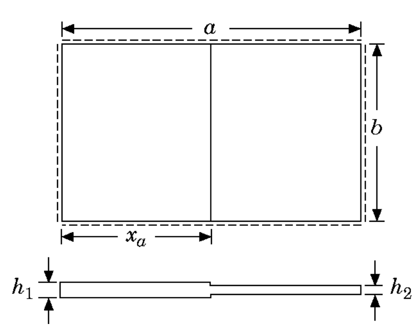

Figure 3. A simply supported stepped plate.

TABLE 5

Frequency parameters λ for square fully simply supported stepped plate with $x_a = 0{\cdot}5$ *and* $\nu = 0{\cdot}3$, $\alpha = h_2/h_1$ ($\lambda^2 = \rho h_1 \omega^2 a^4/D_0$)

α	Solution	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6	Mode 7	Mode 8
1·0	Yuan [14]	19·7392	49·3482	49·3486	78·9574	98·7102	100·117	128·317	129·533
	QEM (2 × 2)	19·7392	49·3480	49·3480	78·9568	98·7106	98·7106	128·317	128·317
	Exact	19·7392	49·3480	49·3480	78·9568	98·6960	98·6960	128·305	128·305
0·9	Yuan	18·7165	46·8060	46·8948	74·9799	93·4028	94·6799	121·595	123·137
	QEM	18·7165	46·8055	46·8948	74·9796	93·3620	93·4170	121·607	121·970
	Exact	18·7165	46·8055	46·8948	74·9795	93·3484	93·4023	121·594	121·958
0·8	Yuan	17·6240	44·1156	44·4416	70·8822	87·3978	88·3631	114·340	116·785
	QEM	17·6240	44·1151	44·4417	70·8819	87·1681	87·4141	114·356	115·649
	Exact	17·6240	44·1151	44·4416	70.8818	87·1557	87·3972	114·339	115·637
0·7	Yuan	16·4834	41·3105	41·8701	66·4611	80·8527	81·2108	106·808	110·137
	QEM	16·4834	41·3101	41·8701	66·4610	80·1672	80·8730	106·832	109·027
	Exact	16·4834	41·3100	41·8700	66·4607	80·1563	80·8522	106·807	109·015
0·6	Yuan	15·3505	38·4398	38·9094	61·3990	73·2279	74·1927	99·4490	102·916
	QEM	15·3505	38·4394	38·9095	61·3992	72·3570	74·2190	99·4872	101·834
	Exact	15·3505	38·4394	38·9094	61·3986	72·3476	74·1923	99·4484	101·821
0·5	Yuan	14·3184	35·1142	35·4469	55·5361	64·1173	68·0257	92·4050	95·3841
	QEM	14·3184	35·1145	35·4466	55·5368	63·4916	68·0631	92·4766	94·3480
	Exact	14·3184	35·1142	35·4465	55·5356	63·4834	68·0253	92·4043	94·3332

with central step ($x_a = 0{\cdot}5$) are based on a 2 by 2 element model. Compared with the exact solution, the QEM shows high accuracy even though the C^2 and C^3 continuity conditions are not explicitly enforced on the geometrically discontinuous interface. However, inter-element compatibility conditions for the QEM are achieved by the accurate continuous moderately high order basis functions in each subdomain, which not only assure C^0 and C^1 accuracy for the field variables but also deliver a good approximation for C^2 and C^3 compatibility for a fourth order equation system. The derivatives of the field variables on the element interfaces can be calculated by approximating them in an average sense from each subdomain.

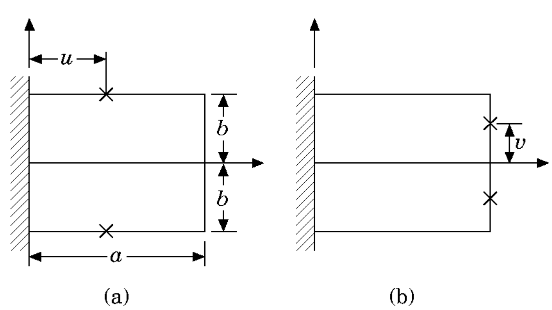

Figure 4. (a) A rectangular plate with symmetrical point supports at the two parallel free edges. (b) A rectangular plate with symmetrical point supports at the free edge opposite to the clamped edge.

TABLE 6

The first five frequency parameters λ of a square cantilever plate ($\lambda^2 = \rho h \omega^2 a^4/D$, $2b/a = 1$, $\nu = 0{\cdot}333$)

(a) Symmetric modes

Mode	QEM (1 × 1)	QEM (2 × 2)	QEM (4 × 4)	Gorman [16]
1	3·405	3·453	3·454	3·459
2	20·52	20·94	21·04	21·09
3	27·06	27·05	27·00	27·06
4	52·46	53·04	53·29	53·53
5	61·71	60·90	60·97	61·12

(b) Antisymmetric modes

Mode	QEM (1 × 1)	QEM (2 × 2)	QEM (4 × 4)	Gorman [16]
1	8·133	8·283	8·339	8·356
2	29·70	30·25	30·45	30·55
3	65·07	63·58	63·61	63·67
4	70·45	69·82	70·40	70·64
5	95·25	92·04	92·11	92·21

4.5. FREE VIBRATION ANALYSIS OF CANTILEVER PLATE AND CANTILEVER PLATE WITH SYMMETRIC POINT SUPPORTS AT THE EDGES

The cantilever plate problem is commonly encountered in engineering structural applications; however, one will find that it is difficult to apply either exact solutions of the Levy type or approximation series type solutions such as Rayleigh–Ritz or DQM to this type of problem because of the free edges. The difficulties and limitations encountered in solving free vibration cantilever plate problems by employing Rayleigh–Ritz and series methods were discussed in, for example, Bassily and Dickinson [15] and Gorman [16]. To

TABLE 7

The first three frequency parameters λ of a square cantilever plate with symmetrical point supports at the two parallel free edges ($\lambda^2 = \rho h \omega^2 a^4/D$, $2b/a = 1$, $u = 0{\cdot}5a$, $\nu = 0{\cdot}333$)

(a) Symmetric modes

Mode	QEM (1 × 1)	QEM (2 × 2)	QEM (4 × 4)	Saliba [17]
1	6·133	6·070	6·072	6·082
2	25·39	25·42	25·37	25·42
3	39·40	38·67	38·56	38·64

(b) Antisymmetric modes

Mode	QEM (1 × 1)	QEM (2 × 2)	QEM (4 × 4)	Saliba [17]
1	16·24	16·03	16·03	16·03
2	51·86	50·65	50·74	50·76
3	69·04	68·33	68·67	68·80

TABLE 8

The first three frequency parameters λ of a square cantilever plate with symmetrical point supports at the edge opposite to the clamped side ($\lambda^2 = \rho h \omega^2 a^4 / D$, $2b/a = 1$, $v = 0{\cdot}5b$, $\nu = 0{\cdot}333$)

(a) Symmetric modes

Mode	QEM (1 × 1)	QEM (2 × 2)	QEM (4 × 4)	Saliba [17]
1	14·19	14·39	14·42	14·44
2	27·00	26·99	26·95	27·02
3	45·22	45·10	45·06	45·17

(b) Antisymmetric modes

Mode	QEM (1 × 1)	QEM (2 × 2)	QEM (4 × 4)	Saliba [17]
1	17·04	17·23	17·29	17·33
2	43·49	43·36	43·39	43·45
3	70·07	68·85	69·24	69·42

investigate the applicability and the accuracy of the QEM for this family of problems, various QEM grid models are obtained and compared with highly accurate analytical solutions proposed by Gorman [16] who exploited a superposition method. Furthermore, free vibrations of rectangular cantilever plates with symmetric point supports at the edges are investigated by the QEM. This family of problems also has analytical solutions obtained by Saliba [17], who essentially extended the application of the superposition method used by Gorman [16].

To formulate these problems, one may consider the rectangular plates shown in Figure 4. Three types of special cases are considered here: a rectangular plate with one edge clamped and the other three edges free, a cantilever plate with symmetric point supports at the parallel free edges as shown in Figure 4(a), and a rectangular plate with symmetric point supports at the free edge opposite to the clamped edge (Figure 4(b)).

Numerical results for these three cases are obtained by using different mesh QEM models. Although all three of these examples can be modelled by taking advantage of symmetry, as in reference [16], general formulations are considered here. Because the QEM uses the Galerkin finite element technique, applying the boundary conditions is quite simple and straightforward.

For the clamped edges, the degrees of freedom in rotation and displacement are constrained in constructing the stiffness matrix. For the symmetric point supports, only the displacements are constrained as the discretized points. Both square plates and rectangular plates of various aspect ratios are investigated. The numerical results for the square plate QEM models are listed in Tables 6, 7 and 8, respectively. Furthermore, the first five mode shapes for the plate with three free edges and the point supported cantilever plate examples are plotted in Figures 5, 6 and 7. The results show excellent mode shape definition despite the rather coarse meshes used (4 × 4). The frequency parameters for rectangular cantilever plates with symmetric point supports and for different aspect ratios are obtained by QEM 4 × 4 mesh models. The results are listed in Tables 9 and 10. All results show excellent comparison with results by Gorman [16] and Saliba [17].

5. CONCLUSIONS

The superior accuracy of the quadrature element method (QEM) as applied to the solutions of free plate vibration problems has been demonstrated in this study through numerical investigations. The plate element employs high order non-conventional displacement interpolations and renders quite satisfactory performance. Because of the use

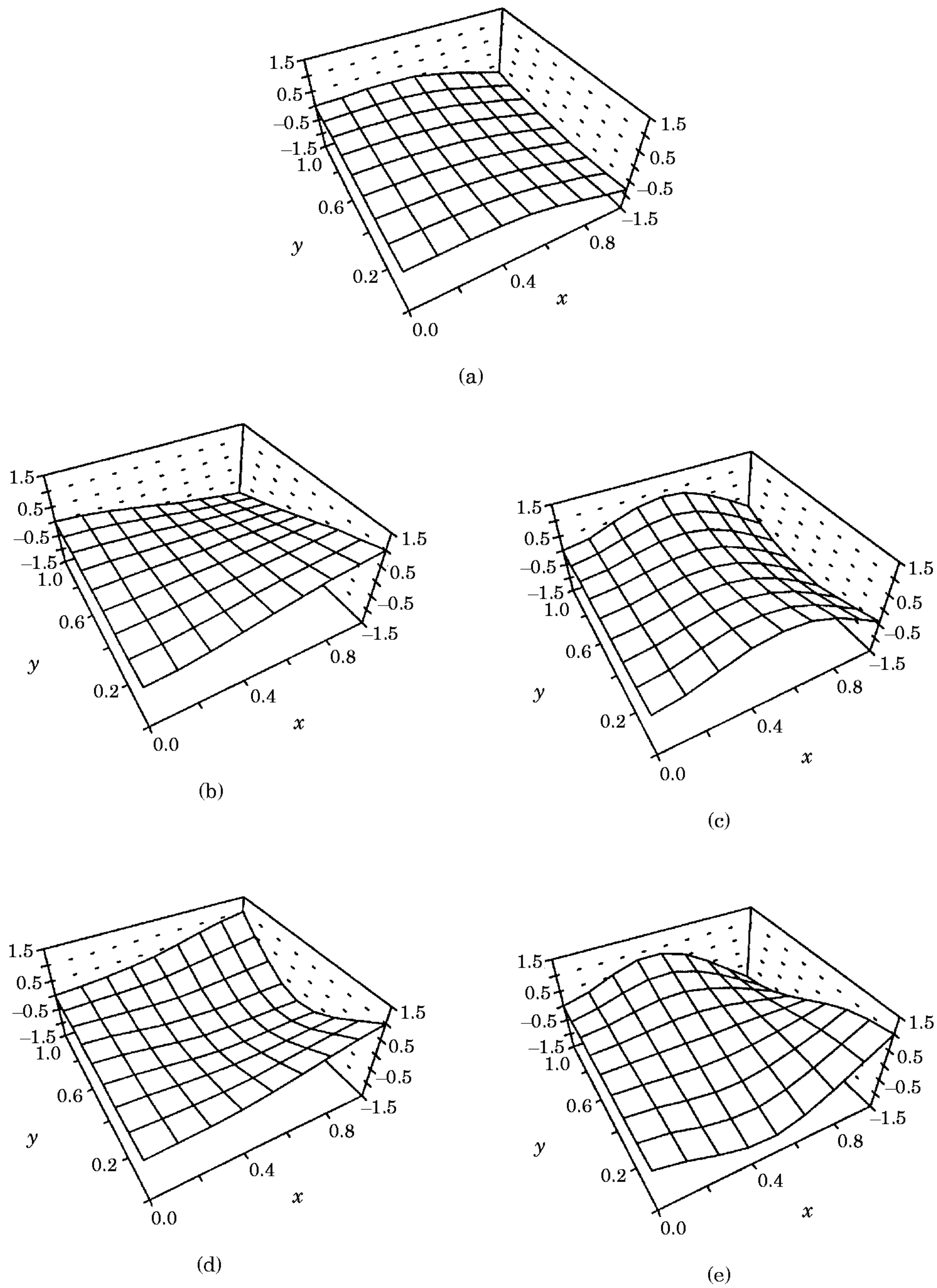

Figure 5. Mode shapes of a square cantilever plate, QEM 4×4: (a) mode 1, $\lambda = 3{\cdot}454$; (b) mode 2, $\lambda = 8{\cdot}339$; (c) mode 3, $\lambda = 21{\cdot}04$; (d) mode 4, $\lambda = 30{\cdot}45$; (e) mode 5, $\lambda = 53{\cdot}29$.

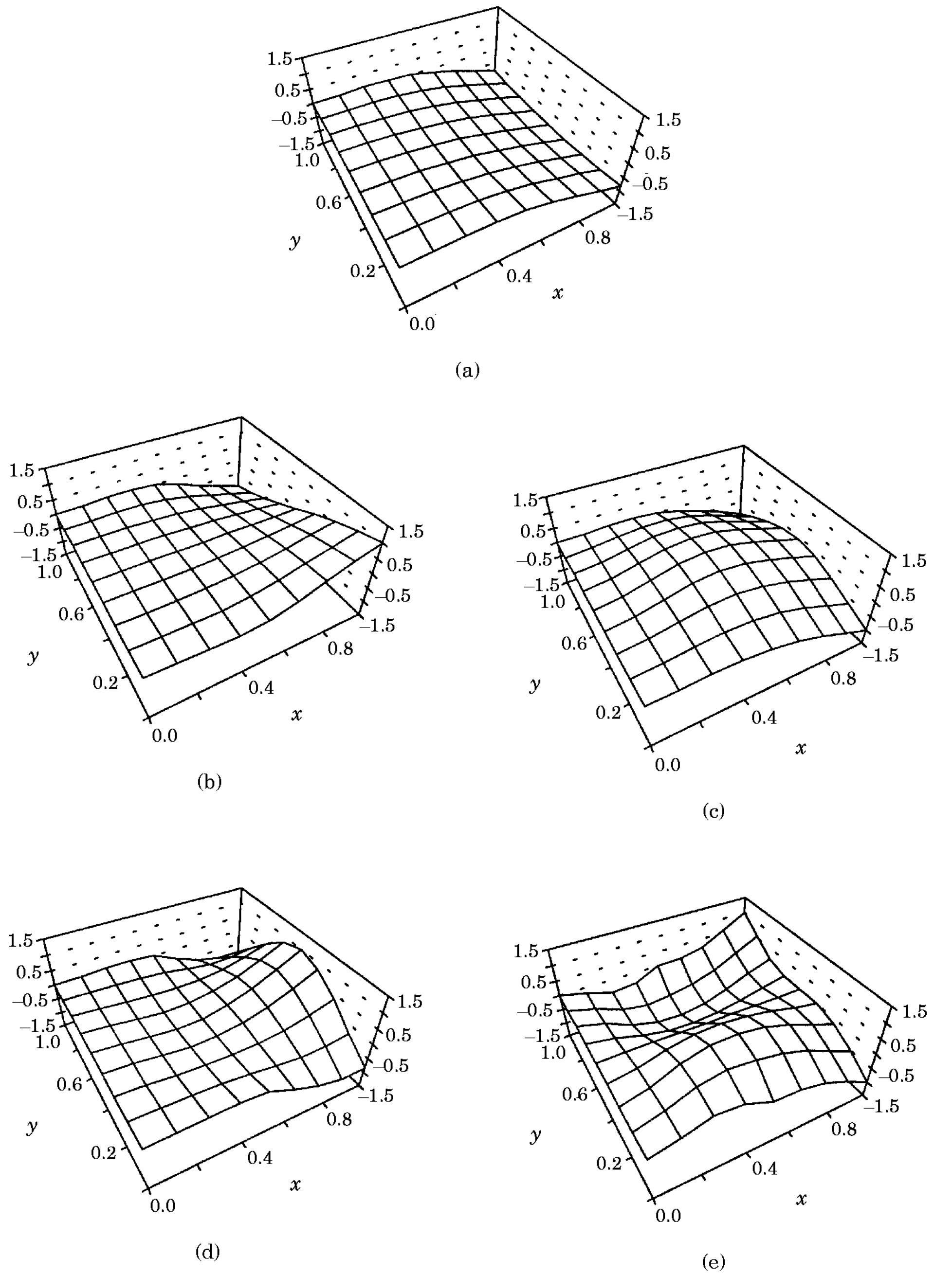

Figure 6. Mode shapes of a square cantilever plate with symmetric point supports at the two parallel free edges, QEM 4 × 4: (a) mode 1, $\lambda = 6{\cdot}072$; (b) mode 2, $\lambda = 16{\cdot}03$; (c) mode 3, $\lambda = 25{\cdot}37$; (d) mode 4, $\lambda = 38{\cdot}56$; (e) mode 5, $\lambda = 50{\cdot}74$.

of a high order approximation, full integration was carried out in calculating the individual element stiffnesses. Therefore, as in the FEM, the frequencies calculated by the QEM will approach the exact values from above. Despite being somewhat different from the conventional serendipity element, all interpolations and quadrature procedures can be presented in an explicit form that is well suited for implementation in a computer code.

The QEM is especially useful for discretization problems with discontinuities in the computational domain or in the boundary conditions. These problems typically yield solutions with large oscillations and unacceptable error convergence for other high order or series type numerical methods that use global basis functions. Here, the present element

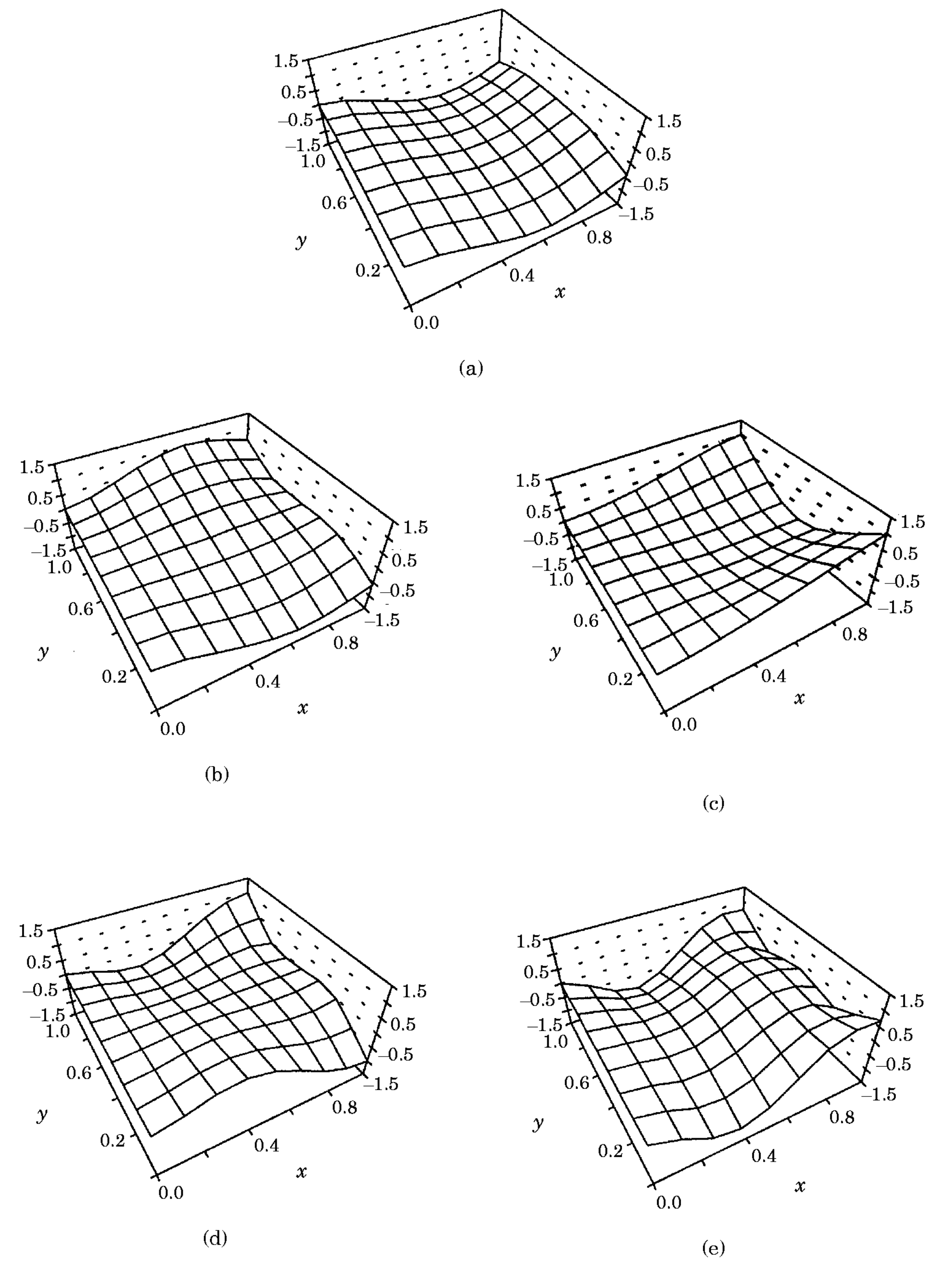

Figure 7. Mode shapes of a square cantilever plate with symmetric point supports at the edge opposite to the clamped edge, QEM 4×4: (a) mode 1, $\lambda = 14{\cdot}42$; (b) mode 2, $\lambda = 17{\cdot}29$; (c) mode 3, $\lambda = 26{\cdot}95$; (d) mode 4, $\lambda = 43{\cdot}39$; (e) mode 5, $\lambda = 45{\cdot}06$.

TABLE 9

The first three frequency parameters λ of rectangular cantilever plates with symmetric point supports at the two parallel free edges ($\lambda^2 = \rho h\omega^2 a^4/D$, $u = 0{\cdot}5a$, $v = 0{\cdot}333$)

(a) Symmetric modes

Mode	Solution	2b/a: 1/3	1/2	2/1	3/1
1	QEM (4 × 4)	8·820	8·207	4·327	3·909
	Saliba [17]	8·820	8·209	4·353	3·939
2	QEM (4 × 4)	59·48	47·70	14·39	9·366
	Saliba [17]	59·48	47·72	14·43	9·406
3	QEM (4 × 4)	66·73	59·72	24·48	19·29
	Saliba [17]	66·75	59·76	24·57	19·31

(b) Antisymmetric modes

Mode	Solution	2b/a: 1/3	1/2	2/1	3/1
1	QEM (4 × 4)	41·52	29·05	8·368	5·842
	Saliba [17]	41·53	29·05	8·384	5·857
2	QEM (4 × 4)	107·5	86·25	21·95	13·97
	Saliba [17]	107·5	86·33	21·96	13·98
3	QEM (4 × 4)	142·4	106·2	33·16	24·97
3	Saliba [17]	142·4	106·2	33·06	24·99

TABLE 10

The first three frequency parameters λ of rectangular cantilever plates with symmetric point supports at the edge opposite to the clamped side ($\lambda^2 = \rho h\omega^2 a^4/D$, $v = 0{\cdot}5b$, $v = 0{\cdot}333$)

(a) Symmetric modes

Mode	Solution	2b/a: 1/3	1/2	2/1	3/1
1	QEM (4 × 4)	14·81	14·82	9·698	6·587
	Saliba [17]	14·81	14·82	9·775	6·693
2	QEM (4 × 4)	47·98	47·69	13·19	9·301
	Saliba [17]	47·99	47·72	13·24	9·369
3	QEM (4 × 4)	100·2	90·96	27·59	20·34
	Saliba [17]	100·2	91·30	27·62	20·34

(b) Antisymmetric modes

Mode	Solution	2b/a: 1/3	1/2	2/1	3/1
1	QEM (4 × 4)	40·17	28·32	10·64	7·340
	Saliba [17]	40·18	28·33	10·73	7·455
2	QEM (4 × 4)	87·72	66·40	23·04	17·28
	Saliba [17]	87·75	66·43	23·06	17·27
3	QEM (4 × 4)	147·3	117·2	33·83	26·07
	Saliba [17]	147·3	117·3	34·23	26·71

method can be used properly to isolate such discontinuities and attain excellent convergence.

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